Likelihood based Goodness-of-fit tests for the Weibull and Extreme Value distributions

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24th January 2013

Introduction

Risk management of industrial facilities, such as EDF's (major French electric utility) power plants, needs to accurately predict system reliability:

- Building of relevant probabilistic models
- Statistical inference of the developed models
- Validation of the fitted models using statistical criteria such as **goodness-of-fit** tests
- Comparison of the different competing models

Problem statement

Let $X_1, \ldots, X_n$ be lifetimes of independent identical non repairable systems

Objective

To find a relevant model for the sample's distribution

Usual models: Exponential and Weibull distributions

Goodness-of-fit tests for the Weibull distribution

Tests based on the empirical distribution function:

- KS
- CM
- AD

Test based on probability plots:

- R^2_EJG
- Z^2
- SPP

Tests based on the normalized spacings:

- MSF
- TS
- LOS

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New likelihood based tests

Goodness-of-fit (GOF) tests

GOF test

Statistical test of H_0: "The sample $X_1, ..., X_n$ comes from $\mathcal{F}$" vs H_1: "The sample $X_1, \ldots, X_n$ does not come from $\mathcal{F}$", where $\mathcal{F}$ is a family of distributions

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In our case $\mathcal{F}$ will be the family of **Weibull** distributions

Principle of the Likelihood based tests

- Embed the tested distribution in a larger parametric family and test a specific value of the parameter of this family
- Three tests: the score, Wald and likelihood ratio tests

Preliminary results and notations

- The Weibull distribution $\mathcal{W}(\eta, \beta)$ is defined by its cumulative distribution function:

$$F(x; \eta, \beta) = 1 - \exp\left(-\left(\frac{x}{\eta}\right)^{\beta}\right), x \geq 0,\ \eta > 0,\ \beta > 0$$

- For all i, the $\ln X_i$ have the extreme value distribution $\mathcal{EV}_1(\ln \eta, 1/\beta)$ with cumulative distribution function

$$G(y; \mu, \sigma) = 1 - e^{-e^{(y-\mu)/\sigma}}, \quad y \in \mathbb{R}$$

where $\mu = \ln \eta$ and $\sigma = 1/\beta > 0$

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- Three methods for estimating the parameters η and β from an i.i.d. sample $X_1, \ldots, X_n$:
    - The maximum likelihood estimators (MLEs) $\hat{\eta}_n$ and $\hat{\beta}_n$
    - The least squares estimators (LSEs) $\widetilde{\eta}_n$ and $\widetilde{\beta}_n$
    - The moment estimators (MEs) $\breve{\eta}_n$ and $\breve{\beta}_n$

Preliminary results and notations

- The MLEs $\hat{\eta}_n$ and $\hat{\beta}_n$ of η and β are solutions of the equations:

$$\left\{ \begin{array}{l} \dfrac{n}{\hat{\beta}_n} + \displaystyle\sum_{i=1}^{n} \ln X_i - \dfrac{n}{\displaystyle\sum_{i=1}^{n} X_i^{\hat{\beta}_n}} \displaystyle\sum_{i=1}^{n} X_i^{\hat{\beta}_n} \ln X_i = 0 \\ \hat{\eta}_n = \left(\dfrac{1}{n} \displaystyle\sum_{i=1}^{n} X_i^{\hat{\beta}_n} \right)^{1/\hat{\beta}_n} \end{array} \right.$$

Preliminary results and notations

- The Weibull probability plot:

$$\left(\ln X_i^*, \ln\left[-\ln\left(1-\tfrac{i}{n}\right)\right]\right),\ i \in \{1,\dots,n-1\}$$

$X_1^* \leq \ldots \leq X_n^*$ are the order statistics of $X_1,\dots,X_n$

- The LSEs $\widetilde{\eta}_n$ and $\widetilde{\beta}_n$ are solutions of the equations:

$$\widetilde{\beta}_n = \frac{\displaystyle\sum_{i=1}^{n}(c_i-\overline{c})^2}{\displaystyle\sum_{i=1}^{n}(\ln X_i - \overline{\ln X})(c_i-\overline{c})} \text{ and } \ln\widetilde{\eta}_n = \overline{\ln X} - \frac{\overline{c}}{\widetilde{\beta}_n}$$

where $c_i = \ln\left[-\ln\left(1-\tfrac{1}{n}(i-0.5)\right)\right],\ i \in \{1,\dots,n\}$

Preliminary results and notations

- The MEs $\breve{\eta}_n$ and $\breve{\beta}_n$ are solutions of the equations:

$$\breve{\beta}_n = \frac{\pi}{\sqrt{6}S} \text{ and } \ln \breve{\eta}_n = \overline{\ln X} + \frac{\gamma_E}{\breve{\beta}_n}$$

$$\text{where } S = \left[\frac{1}{n-1} \sum_{i=1}^{n} (\ln X_i - \overline{\ln X})^2 \right]^{1/2}$$

Preliminary results and notations

- For all i, the $Y_i = \ln\left(\frac{X_i}{\eta}\right)^{\beta}$ have the extreme value distribution $\mathcal{EV}_1(0,1)$

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- For all i, the $Y_i = \ln\left(\frac{X_i}{\eta}\right)^{\beta}$ have the extreme value distribution $\mathcal{EV}_1(0,1)$
- For all i, let $\hat{Y}_i = \ln\left(\frac{X_i}{\hat{\eta}_n}\right)^{\hat{\beta}_n}$, where $\hat{\eta}_n$ and $\hat{\beta}_n$ are the MLE of η and β. The distribution of $(\hat{Y}_1, \dots, \hat{Y}_n)$ does not depend on η and β (Antle and Bain, 1969)

Preliminary results and notations

- For all i, let $\widetilde{Y}_i = \ln\left(\dfrac{X_i}{\widetilde{\eta}_n}\right)^{\widetilde{\beta}_n}$, where $\widetilde{\eta}_n$ and $\widetilde{\beta}_n$ are the least squares estimators based on the WPP. The distribution of $(\widetilde{Y_1}, \ldots, \widetilde{Y_n})$ does not depend on η and β (Liao Shimokawa, 1999)
- For all i, let $\check{Y}_i = \ln\left(\dfrac{X_i}{\check{\eta}_n}\right)^{\check{\beta}_n}$, where $\check{\eta}_n$ and $\check{\beta}_n$ are the moment estimators $\check{\eta}_n$ and $\check{\beta}_n$.The distribution of $(\check{Y}_1, \ldots, \check{Y}_n)$ does not depend on η and β

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- For all i, let $\breve{Y}_i = \ln\left(\dfrac{X_i}{\breve{\eta}_n}\right)^{\breve{\beta}_n}$, where $\breve{\eta}_n$ and $\breve{\beta}_n$ are the moment estimators $\breve{\eta}_n$ and $\breve{\beta}_n$.The distribution of $(\breve{Y}_1, \dots, \breve{Y}_n)$ does not depend on η and β

The fact that the distributions of the samples $\hat{Y}_i$, $\widetilde{Y}_i$ and $\breve{Y}_i$ are independent of η and β allows to build GOF tests statistics as functions of these samples

Reminder

In a previous study, the likelihood based tests for the Exponential distribution have the best performance among several known GOF tests

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Principle of the Likelihood based tests

- Embed the **Weibull** distribution $\mathcal{W}(\eta, \beta)$ in a **Generalized Weibull** parametric family $\mathcal{GW}(\theta, \eta, \beta)$
- Test wheither $\theta = \theta_0$ in the case where $\mathcal{GW}(\theta_0, \eta, \beta) = \mathcal{W}(\eta, \beta)$
- Likelihood based tests: the score, Wald and likelihood ratio tests

Generalized Weibull distributions $\mathcal{GW}$

Name	cdf	Characteristics
Exponentiated Weibull $\mathcal{EW}(\theta, \eta, \beta)$	$F_X(x) = \left[1 - e^{-(x/\eta)^\beta}\right]^\theta$ $\theta, \eta, \beta > 0$	Weibull if $\theta = 1$ DHR if $\beta < 1, \theta < 1$ IHR if $\beta > 1, \theta > 1$ BT or IHR if $\beta > 1, \theta < 1$ UBT or DHR if $\beta < 1, \theta > 1$
Generalized Gamma $\mathcal{GG}(k, \eta, \beta)$	$F_X(x) = \frac{1}{\Gamma(k)} \gamma(k, (x/\eta)^\beta)$ $k, \eta, \beta > 0,$ $\gamma(s, x) = \int_0^x v^{s-1} e^{-v} dv$	Weibull if $k = 1$ if $\frac{1-k\beta}{\beta-1} > 0, \begin{cases} \text{BT if } \beta > 1 \\ \text{UBT if } 0 < \beta < 1 \end{cases}$ otherwise $\begin{cases} \text{IHR if } \beta > 1 \\ \text{DHR if } 0 < \beta < 1 \end{cases}$
Additive Weibull $\mathcal{AW}(\xi, \eta, \beta)$	$F_X(x) = 1 - e^{-\xi x - (\frac{x}{\eta})^\beta}$ $\xi, \eta, \beta > 0$	Weibull if $\xi \to 0$ IHR if $\beta > 1$ DHR if $\beta < 1$

Generalized Weibull distributions $\mathcal{GW}$

Name	cdf	Characteristics
Burr Generalized Weibull $\mathcal{BGW}(\lambda,\eta,\beta)$	$F_X(x) = 1 - \left[1+\lambda(x/\eta)^\beta\right]^{-\frac{1}{\lambda}}$ $\lambda,\eta,\beta > 0$	Weibull if $\lambda \to 0$ DHR if $\beta < 1$ UBT if $\beta > 1$
Marshall-Olkin Extended Weibull $\mathcal{MO}(\alpha,\eta,\beta)$	$F_X(x) = 1 - \frac{\alpha e^{-(x/\eta)^\beta}}{1-(1-\alpha)e^{-(x/\eta)^\beta}}$ $\alpha,\eta,\beta > 0$	Weibull if $\alpha = 1$ IHR if $\alpha \geq 1,\ \beta \geq 1$ DHR if $\alpha \leq 1,\ \beta \leq 1$ other shapes
Modified Weibull $\mathcal{MW}(\rho,\eta,\beta)$	$F_X(x) = 1 - e^{-(\frac{x}{\eta})^\beta e^{\rho x}}$ $\rho,\eta,\beta > 0$	Weibull if $\rho = 0$ IHR if $\beta > 1$ BT if $0 < \beta < 1$
Power Generalized Weibull $\mathcal{PGW}(\nu,\eta,\beta)$	$F_X(x) = 1 - e^{1-\left(1+(x/\eta)^\beta\right)^{\frac{1}{\nu}}}$ $\nu,\eta,\beta > 0$	Weibull if $\nu = 1$ IHR if $\beta > 1$ and $\beta > \nu$ DHR if $0 < \beta < 1$ and $\beta \leq \nu$ BT if $0 < \nu < \beta < 1$ UBT if $\nu > \beta > 1$

The likelihood based GOF tests - Approach 1

Include Weibull $\mathcal{W}(\eta, \beta)$ in a Generalized Weibull distribution $\mathcal{GW}(\boldsymbol{\theta})$ with three parameters $\boldsymbol{\theta} = (\theta, \eta, \beta)$
H_0: "$\theta = \theta_0$" vs "$\theta \neq \theta_0$ $\Leftrightarrow$ H_0: "$X \rightsquigarrow$ Weibull" vs "$X \not\rightsquigarrow$ Weibull"

- Let $\widetilde{\boldsymbol{\theta}_n} = (\theta_0, \widetilde{\eta}_n(\theta_0), \widetilde{\beta}_n(\theta_0))$ where for a given value θ_0 of θ, $(\widetilde{\eta}_n(\theta_0), \widetilde{\beta}_n(\theta_0))$ is the MLE of (η, β)
- The likelihood function for $\boldsymbol{\theta}$ is $L(\boldsymbol{\theta})$
- Let $\hat{\boldsymbol{\theta}}_{\boldsymbol{n}} = (\hat{\theta}_n, \hat{\eta}_n, \hat{\beta}_n) = argmax_{\boldsymbol{\theta}} L(\boldsymbol{\theta})$
- $l(\boldsymbol{\theta}) = \ln L(\boldsymbol{\theta})$ is the log-likelihood function
- The score vector is $U(\boldsymbol{\theta}) = \nabla l(\boldsymbol{\theta})$
- The observed Fisher information matrix is denoted $I(\boldsymbol{\theta})$. Its inverse is denoted:

$$I(\boldsymbol{\theta})^{-1} = \begin{pmatrix} I^{11}(\boldsymbol{\theta}) & I^{12}(\boldsymbol{\theta}) \\ I^{21}(\boldsymbol{\theta}) & I^{22}(\boldsymbol{\theta}) \end{pmatrix}$$

The likelihood based GOF tests - Approach 1

1. Choose a generalized Weibull family $\mathcal{GW}(\theta, \eta, \beta)$.
   Let $f_X(x; \theta, \eta, \beta)$ be its pdf
2. Compute the likelihood $L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f_X(x_i; \theta, \eta, \beta)$ and the MLEs of θ, η and β: $\hat{\theta}_n$, $\hat{\eta}_n$ and $\hat{\beta}_n$
3. Compute the score vector and the observed information 3x3 matrix: $U(\boldsymbol{\theta})$ and $I(\boldsymbol{\theta})$

The likelihood based GOF tests - Approach 1

4. The likelihood based statistics are:
   - Wald: $W = \frac{(\hat{\theta}_n - \theta_0)^2}{I^{11}(\hat{\theta}_n)}$
   - The score: $S = U_1(\widetilde{\theta}_n)^2 I^{11}(\widetilde{\theta}_n)$
   - The likelihood ratio statistic: $LR = -2 \ln \left[\frac{L(\widetilde{\theta}_n)}{L(\hat{\theta}_n)} \right]$

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Approach used in Mudholkar et al (1993, 1996), Bousquet et al (2000) and Caroni (2010)

The likelihood based GOF tests - Approach 1

- Under the null hypothesis H_0, W, S and LR converge to the χ^2_1 distributions when n tends to infinity

This approach presents different drawbacks:

- The MLE of the three parameters distributions is not always easy and it usually requires large samples
- The distributions under H_0 of W, S and LR depend on the parameters in the case of small samples. So, the tests can not be applied to small samples
- The tests in this case are asymptotic. The rejection of Weibul hypothesis is done if the statistics are greater than the quantile of order $(1-\alpha)$ of the χ^2_1 distribution

The likelihood based GOF tests - Approach 2

Include the Weibull distribution in a Generalized Weibull family and deduce the inclusion of the sample $Y_i = \ln(X_i/\eta)^\beta, i = 1, \ldots, n$, that follows the standard type I Extreme Value distribution $\mathcal{EV}_1(0,1)$, in larger families with only one parameter

- The score and Fisher information are uni-dimensional: $I(\theta) = -\frac{\partial^2 l(\theta)}{\partial^2 \theta}$ and $U(\theta) = \frac{\partial l(\theta)}{\partial \theta}$
- The likelihood based statistics are:
  - Wald: $W = I(\theta_0)(\hat{\theta}_n - \theta_0)^2$
  - Score: $S = U^2(\theta_0)/I(\theta_0)$
  - Likelikhood ratio: $LR = -2\ln\left[\frac{L(\theta_0)}{L(\hat{\theta}_n)}\right]$

The likelihood based GOF tests - Approach 2

1. Choose a generalized Weibull family $\mathcal{GW}(\theta, \eta, \beta)$.
   Let $f_X(x; \theta, \eta, \beta)$ be its pdf
2. Compute the pdf of $Y = \ln X$ when $\eta = \beta = 1$:

$$f_Y(y; \theta) = e^y f_X(e^y; \theta, 1, 1)$$

3. Compute the likelihood $L(\theta) = \prod_{i=1}^{n} f_Y(y_i; \theta)$ and the MLE of θ, $\hat{\theta}_n$
4. Compute the score and observed information:

$$U(\theta) = \frac{\partial \ln L(\theta)}{\partial \theta}$$
$$I(\theta) = -\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}$$

The likelihood based GOF tests - Approach 2

5. The likelihood based statistics are:
   - $W = I(\theta_0)(\hat{\theta}_n - \theta_0)^2$
   - $S = \frac{U^2(\theta_0)}{I(\theta_0)}$
   - $LR = -2 \ln \frac{L(\theta_0)}{L(\hat{\theta}_n)}$
6. Replace Y_i by $\hat{Y}_i$. If T denotes a particular $\mathcal{GW}$ model chosen, the corresponding statistics are denoted $\hat{T}_w$, $\hat{T}_s$ and $\hat{T}_l$
7. Do the same thing with $\widetilde{Y}_i$ and $\check{Y}_i$ and derive $\widetilde{T}_w$, $\widetilde{T}_s$, $\widetilde{T}_l$, $\check{T}_w$, $\check{T}_s$ and $\check{T}_l$
8. Reject the Weibull assumption at the significance level α if the statistic is greater than the corresponding quantile of order $1 - \alpha$

Approach - 2: Example of the Exponentiated Weibull family

1. The Generalized Weibull family used is the Exponential Weibull distribution $\mathcal{GW}\ (\theta, \eta, \beta) = \mathcal{EW}(\theta, \eta, \beta)$

Approach - 2: Example of the Exponentiated Weibull family

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3. The null hypothesis H_0: "$\theta = 1$" vs H_1: "$\theta \neq 1$"

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3. The null hypothesis H_0: "$\theta=1$" vs H_1: "$\theta\neq 1$"
4. The score, observed Fisher information and the MLE of θ:

$$U(\theta)=\frac{n}{\theta}+\sum_{i=1}^{n}\ln(1-e^{-e^{Y_i}}),\ I(\theta)=\frac{n}{\theta^2}$$

$$\hat{\theta}_n=-n/\left(\sum_{i=1}^{n}\ln(1-e^{-e^{Y_i}})\right)$$

Approach - 2: Example of the Exponentiated Weibull family

5. The likelihood based statistics are:
   - Wald: $EW_w = I(1)(\hat{\theta}_n - 1)^2 = n(\hat{\theta}_n - 1)^2$
   - Score: $EW_s = U^2(1)/I(1) = n\left(1 - \frac{1}{\hat{\theta}_n}\right)^2$
   - Likelihood ratio: $EW_l = -2\ln\frac{L(1)}{L(\hat{\theta}_n)} = 2n\left(\ln\hat{\theta}_n - 1 + \frac{1}{\hat{\theta}_n}\right)$

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   - Likelihood ratio: $EW_l = -2\ln\frac{L(1)}{L(\hat{\theta}_n)} = 2n\left(\ln\hat{\theta}_n - 1 + \frac{1}{\hat{\theta}_n}\right)$
6. 9 tests statistics: $\widehat{EW}_w$, $\widehat{EW}_s$, $\widehat{EW}_l$, $\widetilde{EW}_w$, $\widetilde{EW}_s$, $\widetilde{EW}_l$, $\breve{EW}_w$, $\breve{EW}_s$ and $\breve{EW}_l$

Remarks

- Rejecting the Weibull assumption at the significance level α if the statistic is greater than the corresponding quantile of order $1-\alpha$
- The quantiles are easily obtained by simulating samples $X_1, \ldots, X_n$ from the $\exp(1)$

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- The suggested tests, unlike the classical ones, are exact tests that can be used for small samples
- In approach - 2, unlike the first approach, the ML estimation is computed for only one parameter instead of three
- For small samples, the distributions of W, S and LR statistics are independent of the Weibull parameters because the ML estimator is computed from the transformed samples $\hat{Y}_i$, $\widetilde{Y}_i$ and $\check{Y}_i, i = 1, \dots, n$ that are independent of Weibull parameters whatever the sample size is

Simulations

- 50000 simulated samples of size $n \in \{5, 10, 20, 50\}$
- $\alpha = 5\%$ is the significance level of all the tests
- Alternate distributions studied:
  - Increasing hazard rate **IHR**: $\mathcal{G}(3)$, $\mathcal{AW}2$, $\mathcal{EW}3$
  - Decreasing hazard rate **DHR**: $\mathcal{G}(0.5)$, $\mathcal{AW}1$, $\mathcal{EW}4$
  - Bathtub shaped hazard rate **BT**: $\mathcal{EW}1$, $\mathcal{GG}1$, $\mathcal{PGW}1$, $\mathcal{GG}3$
  - Upside-down bathtub shaped hazard rate **UBT**: $\mathcal{LN}(0.8)$, $\mathcal{IG}(3)$, $\mathcal{EW}2$, $\mathcal{GG}2$, $\mathcal{PGW}2$
- Comparison of the best of these tests with two usual GOF tests for the Weibull distribution: Anderson-Darling *AD* and Tiku-Singh *TS*

Power results for the tests based on the $\mathcal{EW}$, $n = 50$

altern.	$\widehat{EW}_w$	$\widehat{EW}_s$	$\widehat{EW}_l$	$\widetilde{EW}_w$	$\widetilde{EW}_s$	$\widetilde{EW}_l$	$\breve{EW}_w$	$\breve{EW}_s$	$\breve{EW}_l$	% rejection
exp(1)	5	5.1	5.1	5	5	5	5.1	5.1	5.1	5
$\mathcal{W}$(0.5)	4.9	5	5	5.1	5.2	5.1	4.9	4.9	4.9	5
$\mathcal{W}$(3)	5	5	5	5.1	5.1	5.1	5.1	5	5	5
$\mathcal{G}$(3)	20	17	18.1	11.6	12.9	12.4	9.9	11.3	10.7	13.8
$\mathcal{AW}$(2)	81.8	83.4	83	80.2	79.2	79.4	81	80.1	80.4	80.9
$\mathcal{EW}$(3)	53	48.5	50.2	23.7	25.8	25	31.4	34.4	33.3	36.2
$\mathcal{G}$(0.5)	14.6	17.4	16.6	11.7	10.9	11	11.9	11.1	11.3	12.9
$\mathcal{AW}$(1)	99.7	99.8	99.8	55.4	53.4	53.8	70.9	68.5	69.3	74.5
$\mathcal{EW}$(4)	41	46.9	45.2	1.9	1.6	1.6	2.3	1.9	2	16
$\mathcal{EW}$(1)	40.6	46.6	44.9	1.8	1.5	1.6	2.3	1.9	2	15.9
$\mathcal{GG}$(1)	69.5	73.6	72.4	29.9	28.3	28.7	31.1	29.3	29.8	43.6
$\mathcal{PGW}$(1)	23.9	27.7	26.6	14.9	13.9	14.2	14.9	13.9	14.1	18.2
$\mathcal{GG}$(3)	51.5	56.4	55	24.9	23.4	23.7	25.2	23.7	24.1	34.2
$\mathcal{LN}$(0.8)	68.5	64.3	65.9	56.6	59.4	59.3	49.2	52.7	51.3	58.6
$\mathcal{IG}$(3)	94.6	93.2	93.8	95	95.7	95.5	88	89.7	89.1	92.7
$\mathcal{EW}$(2)	38.3	33.8	35.7	23.2	25.4	24.6	20	22.4	21.5	27.2
$\mathcal{GG}$(2)	41.2	36.9	38.6	27.4	29.8	28.9	22.9	25.6	24.6	30.7
$\mathcal{PGW}$(2)	66.5	61.9	63.5	53.8	56.3	55.4	48	51.2	49.8	56.3

Power results for the tests based on the $\mathcal{EW}$, $n = 20$

altern.	$\widehat{EW}_w$	$\widehat{EW}_s$	$\widehat{EW}_l$	$\widetilde{EW}_w$	$\widetilde{EW}_s$	$\widetilde{EW}_l$	$\check{EW}_w$	$\check{EW}_s$	$\check{EW}_l$	% rejection
exp(1)	5	5	4.9	5	4.9	5	5	5	5	5
$\mathcal{W}(0.5)$	5.3	5.3	5.3	5.2	5.1	5.1	5	4.9	5	5.1
$\mathcal{W}(3)$	5	5	5	5.1	5	5	5	5	5	5
$\mathcal{G}(3)$	9.7	7.2	8	5.1	5.9	5.7	5	6	5.6	6.5
$\mathcal{AW}(2)$	49.9	53.7	52.5	46.4	44.5	45.3	48.4	46.5	47.2	48.3
$\mathcal{EW}(3)$	21.5	16.6	18.2	10.9	12.9	12.3	10.4	12.5	11.7	14.1
$\mathcal{G}(0.5)$	8.6	10.8	10	9.2	8.5	8.8	8.7	8	8.2	9
$\mathcal{AW}(1)$	79.8	84.1	82.7	32.7	30.3	31.3	41.4	38.6	39.5	51.2
$\mathcal{EW}(4)$	13.7	18.2	16.6	4.6	4	4.2	4.3	3.7	3.9	8.2
$\mathcal{EW}(1)$	13.5	18	16.5	4.6	4	4.2	4.3	3.7	3.9	8.1
$\mathcal{GG}(1)$	29.2	34.8	33	18	16.5	17.1	17.5	16	16.5	22.1
$\mathcal{PGW}(1)$	11.2	14.1	13.1	11	10.1	10.5	10.2	9.4	9.6	11
$\mathcal{GG}(3)$	21.4	26	24.5	15.9	14.6	15.2	15	13.5	13.9	17.8
$\mathcal{LN}(0.8)$	29.8	23.8	25.8	16.5	19.3	18.5	15	18	16.9	20.4
$\mathcal{IG}(3)$	56.2	49.9	52.3	44.9	48.9	48.9	36.5	41.3	39.9	46.5
$\mathcal{EW}(2)$	15.7	11.9	13.2	7.6	9	8.6	7.2	8.9	8.3	10.1
$\mathcal{GG}(2)$	16.9	12.6	14.1	8.3	9.9	9.4	8	9.7	9.1	10.9
$\mathcal{PGW}(2)$	28.7	22.7	24.2	16.7	19.3	18.5	16.1	19	18	20.4

Comparison with usual GOF tests, $n = 50$

altern.	$\widehat{GG}^1_w$	$\widehat{GG}^1_s$	$\widehat{GG}^1_l$	$\widehat{GG}^2_l$	$\widehat{MW}_w$	$\widehat{PGW}_w$	$\widehat{PGW}_s$	$\widehat{PGW}_l$	$P\breve{G}W_w$	AD	TS
exp(1)	5.1	5.1	5.1	5.5	5	4.9	4.9	5	5	5.6	4.9
$\mathcal{W}(0.5)$	5.1	5	5	5.6	5	5	5	5	5	5.4	5
$\mathcal{W}(3)$	5.1	5	5	5.3	5.3	5	5	5.1	4.9	5.3	5.1
$\mathcal{G}(3)$	18.2	16.8	17.2	21.1	0.4	18.6	15.6	16.7	28.9	14.6	18.9
$\mathcal{AW}(2)$	83.7	84.1	83.9	82.3	81.1	80.6	82.2	81.8	0	72.2	82.2
$\mathcal{EW}(3)$	50.7	49	49.6	56.3	0	49.6	44.8	46.7	66.8	40.8	55.2
$\mathcal{G}(0.5)$	16.8	17.6	17.2	16.7	24.3	16.1	18.6	17.7	0.5	13.5	15.5
$\mathcal{AW}(1)$	99.8	99.8	99.8	99.8	100	99.9	99.9	99.9	0	99.9	99.6
$\mathcal{EW}(4)$	44.1	46.2	45.5	47.4	78.8	52.2	57	55.8	0	57.9	49.4
$\mathcal{EW}(1)$	43.7	45.3	44.6	47.5	78.9	51.8	56.6	55.4	0	58.1	49.8
$\mathcal{GG}(1)$	71.7	73.4	72.9	73.3	89.9	75.3	78.4	77.6	0	69.4	74.9
$\mathcal{PGW}(1)$	26.9	28.4	27.9	27	40.1	26.9	30.2	29.3	0.2	21.1	27.2
$\mathcal{GG}(3)$	54.9	56.7	56.2	55.8	73.2	56.3	60.6	59.4	0	48.3	56.2
$\mathcal{LN}(0.8)$	66.9	65.3	65.8	72.5	0	65.5	60.9	62.7	82.5	56.5	72
$\mathcal{IG}(3)$	94.2	93.6	93.7	96.2	0	93.3	91.6	92.4	98.6	92.3	96.9
$\mathcal{EW}(2)$	35.8	33.7	34.3	40.6	0	35.8	31.4	33.2	51.2	27.9	38.9
$\mathcal{GG}(2)$	38.8	37.1	37.7	44.1	0	39	34.5	36.3	55.6	30.1	42.9
$\mathcal{PGW}(2)$	64.6	62.5	63.1	69.9	0	63	58.1	59.9	79.7	56.9	71.6

Comparison with usual GOF tests, $n = 20$

altern.	$\widehat{GG}^1_w$	$\widehat{GG}^1_s$	$\widehat{GG}^1_l$	$\widehat{GG}^2_l$	$\widehat{MW}_w$	$\widehat{PGW}_w$	$\widehat{PGW}_s$	$\widehat{PGW}_l$	$\check{PGW}_w$	AD	TS
exp(1)	5.1	5.1	5	5.7	5.1	4.8	4.8	4.8	5	5.6	5.1
$\mathcal{W}(0.5)$	5.1	5.1	5.1	5.5	5.1	4.9	4.9	4.8	5.1	5.6	5.2
$\mathcal{W}(3)$	5	5	5	5.6	5	5	4.9	5	5	5.6	5
$\mathcal{G}(3)$	7.9	6.9	7.2	10.4	1.2	8.1	5.7	6.6	15.4	8.5	8.7
$\mathcal{AW}(2)$	53	54.1	53.7	49.6	53.4	46.7	52.7	51.9	0.8	42.1	49.5
$\mathcal{EW}(3)$	17.9	16.2	16.7	22.4	0.3	18.4	13.6	15.3	32.2	16.5	19.9
$\mathcal{G}(0.5)$	10	10.7	10.4	9.4	14.3	9.3	11.2	10.7	1.5	8.8	9.2
$\mathcal{AW}(1)$	81.5	83.3	82.7	82.3	95	85.8	88.7	88	0	89.7	87
$\mathcal{EW}(4)$	15.9	18	17.3	15.3	35.2	18.5	22.8	21.7	0	23.3	18.1
$\mathcal{EW}(1)$	16.3	17.9	17.3	16.9	35	17.8	22.4	21.2	0.1	23.7	17.6
$\mathcal{GG}(1)$	32.2	34.5	33.8	30.7	49.5	33.7	38.7	37.3	0	31.7	34.1
$\mathcal{PGW}(1)$	13.2	14.5	14.1	11.6	19.7	12.8	15.5	14.7	0.7	11.7	12.6
$\mathcal{GG}(3)$	24.1	26.1	25.4	21.9	35.9	24.1	28.2	27.1	0.2	21.6	24
$\mathcal{LN}(0.8)$	25.3	22.8	23.5	30.5	0.1	25.7	19.7	21.9	42.7	22.8	28.8
$\mathcal{IG}(3)$	52.1	48.9	49.8	60.9	0	51.6	43.6	46.8	71.1	50.5	59.7
$\mathcal{EW}(2)$	13	11.7	12.2	15.7	0.6	13.5	9.6	10.9	24.1	12.1	14.4
$\mathcal{GG}(2)$	13.5	12.1	12.5	17.1	0.5	14.5	10.3	11.8	25.9	12.9	15.7
$\mathcal{PGW}(2)$	23.9	21.7	22.5	29.8	0.2	24.9	18.8	21.1	41.1	23.2	28.6

Results and discussion

- The performance of the tests is tightly linked to the hazard rate's shape of the tested alternate
- Some tests are non-consistent for some kinds of alternatives
- Generally, we recommend
  - For IHR alternates: $\widehat{GG}_l^2$
  - For DHR and BT alternates: $\widehat{MW}_w$
  - For UBT alternates: $P\breve{G}W_w$